Mathematics for Electrical Engineering Homework # 3

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Problme 3.1 Find the inverse Laplace transforms $f(t)$ of the following $F(s)$:
(i) $F(s) = \dfrac{s+12}{s^2+9s+14}$
(iii) $F(s) = \dfrac{3s^2+20s+39}{s^2+6s+13}$
(ii) $F(s) = \dfrac{s+4}{s^3+3s^2+12s+10}$

Problme 3.2 Solve the following differential equations by Laplace and inverse Laplace transforms.

$$\text{(i)} \quad \frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x(t) = 2e^{-3t},\ x(0) = 0,\ x'(0) = 1$$

$$\text{(ii)} \quad \frac{d^2x}{dt^2} + 9x(t) = \cos 2t,\ x(0) = 1,\ x'(0) = 0$$

SOLUTIONS
Problem 3.1
(i) We have

$$\begin{aligned} F(s) &= \frac{s+12}{s^2+9s+14} \\ &= \frac{s+12}{(s+2)(s+7)} \\ &= \frac{2}{s+2} - \frac{1}{s+7}. \end{aligned}$$

Thus we obtain

$$f(t) = \mathcal{L}[F(s)] = (2e^{-2t} - e^{-7t})u(t).$$

(ii)

$$\begin{aligned} F(s) &= \frac{3s^2+20s+39}{s^2+6s+13} \\ &= 3 + \frac{2s}{s^2+6s+13} \\ &= 3 + \frac{2s}{(s+3)^2+4} \\ &= 3 + 2\frac{(s+3)}{(s+3)^2+2^2} - 3\frac{2}{(s+2)^2+2^2}. \end{aligned}$$

Thus we obtain

$$f(t) = \mathcal{L}[F(s)] = 3\delta(t) + e^{-3t}(2\cos 2t - 3\sin 2t)u(t).$$

(iii)

$$\begin{aligned}
F(s) &= \frac{s+4}{s^3+3s^2+12s+10} \\
&= \frac{s+4}{(s+1)(s^2+2s+10)} \\
&= \frac{1}{3}\frac{1}{s+1} + \frac{-1+j}{6}\frac{1}{s+1+3j} - \frac{1+j}{6}\frac{1}{s+1-3j} \\
&= \frac{1}{3}\frac{1}{s+1} + \frac{1}{3}\frac{-s+2}{s^2+2s+10} \\
&= \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\frac{s+1}{(s+1)^2+3^2} + \frac{1}{3}\frac{3}{(s+1)^2+3^2}.
\end{aligned}$$

Thus we obtain

$$f(t) = \mathcal{L}[F(s)] = \frac{1}{3}e^{-t}(1-\cos 3t + \sin 3t)u(t).$$

Problem 3.2

(i)Take Laplace transforms of both sides of the equation with the initial conditions:

$$\begin{aligned}
s^2X(s) - sx(0) - x'(0) + 6(sX(s) - x(0)) + 8X(s) &= \frac{2}{s+3} \\
(s^2+6s+8)X(s) &= 1 + \frac{2}{s+3} \\
\Longrightarrow X(s) = \frac{1}{(s+2)(s+4)} &+ \frac{2}{(s+2)(s+3)(s+4)}.
\end{aligned}$$

Thus we have

$$X(s) = \frac{1}{2}\left(\frac{1}{s+2} - \frac{1}{s+4}\right) - \left(\frac{2}{s+2} + \frac{2}{s+3} - \frac{1}{s+4}\right).$$

Taking inverse Laplace transform, we get

$$\begin{aligned}
x(t) &= \frac{1}{2}(e^{-2t} - e^{-4t})u(t) - (2e^{-2t} + 2e^{-3t} - e^{-4t})u(t) \\
&= (\frac{3}{2}e^{-2t} - 2e^{-3t} + \frac{1}{2}e^{-4t})u(t).
\end{aligned}$$

(ii) Take Laplace transforms of both sides of the equations with the initial conditions:

$$\begin{aligned}
s^2X(s) - sx(0) - x'(0) + 9X(s) &= \frac{s}{s^2+4} \\
(s^2+9)X(s) &= s + \frac{s}{s^2+4} \\
\Longrightarrow X(s) &= \frac{s}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}.
\end{aligned}$$

Thus we have

$$X(s) = \frac{s}{s^2+9} + \frac{1}{5}\frac{s}{s^2+4} - \frac{1}{5}\frac{s}{s^2+9}.$$

Hence

$$\begin{aligned}
x(t) &= (\cos 3t + \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t)u(t) \\
&= \frac{1}{5}(\cos 2t + 4\cos 3t)u(t).
\end{aligned}$$