Mathematics for Electrical Engineering Homework # 4

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Problem 4.1 For the following matrices,
i) diagonalize the following matrices if possible. Otherwise, transform them into the Jordan form.
ii) calculate e^{At} and e^{Bt} by using the results in (i).

$$\text{a) } A = \begin{bmatrix} 2 & 1 & -3 \\ -2 & 5 & -2 \\ 1 & -1 & 6 \end{bmatrix}, \quad \text{b) } B = \begin{bmatrix} 1 & -1 & 1 \\ -7 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

Problem 4.2 Given

$$\dot{x}(t) = \begin{bmatrix} 4 & 3 \\ 2 & -1 \end{bmatrix} x(t).$$

i) Calculate e^{At} using Laplace transform.
ii) Let the initial condition be given by

$$x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

Then solve the above differential equation(i.e., calculate $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ with the above initial condition.).

Solutions
Problem 4.1 ia) The characteristic polynomial is given by

$$\begin{aligned} |sI - A| &= \begin{vmatrix} s-2 & -1 & 3 \\ 2 & s-5 & 2 \\ -1 & 1 & s-6 \end{vmatrix} \\ &= (s-2)(s-5)(s-6) + 6 + 2 + 3(s-5) + 2(s-6) - 2(s-2) \\ &= (s-5)[(s^2 - 8s + 12) + 3] + 8 - 12 + 4 \\ &= (s-5)(s^2 - 8s + 15) \\ &= (s-3)(s-5)^2. \end{aligned}$$

Hence eigenvalues are $\lambda = 3, 5$ and the associated eigenvectors are

$$\lambda_1 = 3 : (\lambda_1 I - A)v_1 = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -2 & 2 \\ -1 & 1 & -3 \end{bmatrix} v_1 = 0 \Longrightarrow v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\lambda_2 = 5 : (\lambda_2 I - A)v_2 = \begin{bmatrix} 3 & -1 & 3 \\ 2 & 0 & 2 \\ -1 & 1 & -1 \end{bmatrix} v_2 = 0 \Longrightarrow v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 5 : (\lambda_2 I - A)v_3 = \begin{bmatrix} 3 & -1 & 3 \\ 2 & 0 & 2 \\ -1 & 1 & -1 \end{bmatrix} v_3 = -v_2 \Longrightarrow v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

It follows that the transformation matrix T is given by

$$T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

and we have

$$\begin{aligned}
T^{-1}AT &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ -2 & 5 & -2 \\ 1 & -1 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 0 & 3 \\ -1 & 1 & -6 \\ -5 & 5 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}.
\end{aligned}$$

ib) We calcualte

$$\begin{aligned}
e^{At} &= T \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{5t} & te^{5t} \\ 0 & 0 & e^{5t} \end{bmatrix} T^{-1} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{5t} & te^{5t} \\ 0 & 0 & e^{5t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t} & e^{5t} & te^{5t} \\ e^{3t} & 0 & e^{5t} \\ 0 & -e^{5t} & -te^{5t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t} - te^{5t} & te^{5t} & e^{3t} - (1+t)e^{5t} \\ e^{3t} - e^{5t} & e^{5t} & e^{3t} - e^{5t} \\ te^{5t} & -e^{5t} & (1+t)e^{5t} \end{bmatrix}.
\end{aligned}$$

iia) The characteristic polynomial is given by

$$\begin{aligned}
|sI - B| &= \begin{vmatrix} s-1 & 1 & -1 \\ 7 & s-2 & -1 \\ -2 & -1 & s-2 \end{vmatrix} \\
&= (s-1)(s-2)^2 + 7 + 2 - 2(s-2) - 7(s-2) - (s-1) \\
&= s^3 - 5s^2 - 2s + 24 \\
&= (s+2)(s-3)(s-4).
\end{aligned}$$

Hence eigenvalues are $\lambda = -2, 3, 4$ and the associated eigenvectors are

$$\lambda_1 = -2 : (\lambda_1 I - A)v_1 = \begin{bmatrix} -3 & 1 & -1 \\ 7 & -4 & -1 \\ -2 & -1 & -4 \end{bmatrix} v_1 = 0 \Longrightarrow v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 3 : (\lambda_2 I - A)v_2 = \begin{bmatrix} 2 & 1 & -1 \\ 7 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix} v_2 = 0 \Longrightarrow v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 4 : (\lambda_3 I - A)v_3 = \begin{bmatrix} 3 & 1 & -1 \\ 7 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix} v_3 = 0 \Longrightarrow v_3 = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}.$$

The transformation matrix T is given by

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -4 \\ -1 & 1 & -1 \end{bmatrix}.$$

We calculate

$$\begin{aligned} T^{-1}BT &= \frac{1}{6}\begin{bmatrix} 3 & 1 & -1 \\ 6 & 0 & 6 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ -7 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -4 \\ -1 & 1 & -1 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} -3 & -1 & 1 \\ 9 & 0 & 9 \\ 6 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -4 \\ -1 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}. \end{aligned}$$

iib) We calculate

$$\begin{aligned} e^{Bt} &= T\begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix}T^{-1} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -4 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix}\frac{1}{6}\begin{bmatrix} 3 & 1 & -1 \\ 6 & 0 & 6 \\ 3 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-2t} & 0 & e^{4t} \\ 2e^{-2t} & e^{3t} & -4e^{4t} \\ -e^{2t} & e^{3t} & -e^{4t} \end{bmatrix}\frac{1}{6}\begin{bmatrix} 3 & 1 & -1 \\ 6 & 0 & 6 \\ 3 & -1 & 1 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 3(e^{-2t}+e^{4t}) & e^{-2t}-e^{4t} & -e^{-2t}+e^{4t} \\ 6(e^{-2t}+e^{3t}-2e^{4t}) & 2(e^{-2t}+2e^{4t}) & 2(-e^{-2t}+3e^{3t}-2e^{4t}) \\ 3(-e^{-2t}+2e^{3t}-e^{4t}) & -e^{-2t}+e^{4t} & e^{-2t}+6e^{3t}-e^{4t} \end{bmatrix}. \end{aligned}$$

Problem 4.2 i) Let

$$A = \begin{bmatrix} 4 & 3 \\ 2 & -1 \end{bmatrix}.$$

Since

$$\begin{aligned} |sI - A| &= \begin{vmatrix} s-4 & -3 \\ -2 & s+1 \end{vmatrix} \\ &= (s-4)(s+1) - 6 \\ &= (s+2)(s-5), \end{aligned}$$

we have

$$\begin{aligned} (sI-A)^{-1} &= \frac{1}{(s+2)(s-5)}\begin{bmatrix} s+1 & 3 \\ 2 & s-4 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{s+1}{(s+2)(s-5)} & \dfrac{3}{(s+2)(s-5)} \\ \dfrac{2}{(s+2)(s-5)} & \dfrac{s-4}{(s+2)(s-5)} \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} \dfrac{1}{s+2}+\dfrac{6}{s-5} & -\dfrac{3}{s+2}+\dfrac{3}{s-5} \\ -\dfrac{2}{s+2}+\dfrac{2}{s-5} & \dfrac{6}{s+2}+\dfrac{1}{s-5} \end{bmatrix} \end{aligned}$$

Hence we have

$$\begin{aligned} e^{At} &= \mathcal{L}^{-1}[(sI-A)^{-1}] \\ &= \frac{1}{7}\begin{bmatrix} e^{-2t}+6e^{5t} & -3e^{-2t}+3e^{5t} \\ -2e^{-2t}+2e^{5t} & 6e^{-2t}+e^{5t} \end{bmatrix}. \end{aligned}$$

iii) It follow from ii) that

$$\begin{aligned} x(t) &= e^{At}x(0) \\ &= \frac{1}{7}\begin{bmatrix} e^{-2t}+6e^{5t} & -3e^{-2t}+3e^{5t} \\ -2e^{-2t}+2e^{5t} & 6e^{-2t}+e^{5t} \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 5e^{-2t}+9e^{5t} \\ -10e^{-2t}+3e^{5t} \end{bmatrix}. \end{aligned}$$

Remark:

Check your answers! We know

$$\frac{d}{dt}(e^{At}) = Ae^{At}. \tag{1}$$

For

$$e^{At} = \frac{1}{7}\begin{bmatrix} e^{-2t}+6e^{5t} & -3e^{-2t}+3e^{5t} \\ -2e^{-2t}+2e^{5t} & 6e^{-2t}+e^{5t} \end{bmatrix},$$

we calculate

$$\frac{d}{dt}(e^{At}) = \frac{1}{7}\begin{bmatrix} -2e^{-2t}+30e^{5t} & 6e^{-2t}+15e^{5t} \\ 4e^{-2t}+10e^{5t} & -12e^{-2t}+5e^{5t} \end{bmatrix}.$$

On the other hand, we have

$$Ae^{At} = \begin{bmatrix} 4 & 3 \\ 2 & -1 \end{bmatrix}\frac{1}{7}\begin{bmatrix} e^{-2t}+6e^{5t} & -3e^{-2t}+3e^{5t} \\ -2e^{-2t}+2e^{5t} & 6e^{-2t}+e^{5t} \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -2e^{-2t}+30e^{5t} & 6e^{-2t}+15e^{5t} \\ 4e^{-2t}+10e^{5t} & -12e^{-2t}+5e^{5t} \end{bmatrix}.$$

We have checked that (1) holds and hence e^{At} in this problem is correct.

Next, we have

$$\dot{x}(t) = \frac{1}{7}\begin{bmatrix} -10e^{-2t}+45e^{5t} \\ 20e^{-2t}+15e^{5t} \end{bmatrix},$$

and

$$Ax(t) = \begin{bmatrix} 4 & 3 \\ 2 & -1 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 5e^{-2t}+9e^{5t} \\ -10e^{-2t}+3e^{5t} \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -10e^{2t}+45e^{5t} \\ 20e^{2t}+15e^{5t} \end{bmatrix}.$$

Therefore, we have $\dot{x}(t) = Ax(t)$, which implies the answer is correct.