Problems on Laplace Transforms

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1 Delta and Step Functions

Problem 1.1 Calculate the following integrals.

$$\begin{aligned}
\text{(i) } y(t) &= \int_{-\infty}^{\infty} e^{-2\sigma} u(\sigma - 3) d\sigma, \\
\text{(ii) } y(t) &= \int_{-\infty}^{\infty} e^{-3\sigma} \sigma^2 u(\sigma + 4) d\sigma, \\
\text{(iii) } y(t) &= \int_{-3}^{\infty} \delta(t - \sigma) e^{-2\sigma} d\sigma, \\
\text{(iv) } y(t) &= \int_{-2}^{\infty} \delta(t - \sigma) \sin(5\sigma) d\sigma, \\
\text{(v) } y(t) &= \int_{-\infty}^{\infty} [\delta(t - \sigma) - e^{-(t-\sigma)} u(t - \sigma)] e^{-2\sigma} u(\sigma) d\sigma, \\
\text{(vi) } y(t) &= \int_{-\infty}^{\infty} 2(t - \sigma) e^{-2(t-\sigma)} u(t - \sigma) \sigma^2 u(\sigma) d\sigma.
\end{aligned}$$

2 Laplace Transforms

Problem 2.2 Let $f(t) = \sin t$. Then obtain the Laplace transforms of the following functions.

$$\text{(i) } f(t - t_0), \quad \text{(ii) } f(t - t_0)U(t), \quad \text{(iii) } f(t)U(t - t_0), \quad \text{(iv) } f(t - t_0)U(t - t_0).$$

Problem 2.3 Let $f(t)$ be a periodic function with period T. Then verify that the Laplace transform of $f(t)$ is given by

$$L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt.$$

Problem 2.4 Find the inverse Laplace transforms of the following functions.

$$\text{(i) } L[f(t)] = \frac{1}{(s^2 + a^2)^2}, \quad \text{(ii) } L[f(t)] = \frac{s}{s^4 + 5s^2 + 4}, \quad \text{(iii) } L[f(t)] = \frac{1}{s^3 + 2s^2 + 10s},$$

$$\text{(iv) } L[f(t)] = \frac{s^2 + 10s + 19}{s^2 + 5s + 6}, \quad \text{(v) } L[f(t)] = \frac{3s^2 + 2s + 1}{s^2(s + 1)}, \quad \text{(vi) } L[f(t)] = \frac{2s}{s^3 + 2s^2 - 1}.$$

Problem 2.5 Solve the following differential equations by using the Laplace transforms.

$$\text{(i) } \frac{d^2 y}{dt^2} + 4y(t) = e^{-t}, \ t > 0, \ y(0) = 0, \ y'(0) = 1,$$

(ii) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 7y(t) = t,\ t > 0,\ y(0) = 0,\ y'(0) = 1.$

Problem 2.6 Find the impulse response of the system described by the following differential equation.

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = 2\frac{dx}{dt} + x(t),$$

Problem 2.7 Suppose that the input-output representation is given by the following differential equation.

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = x(t),\ t > 0,\ y(0) = 0,\ y'(0) = 1.$$

(i) Find the impulse response of the system. Calculate the response when the input is $x(t) = U(t-2)$.
(ii) Suppose that the input is given by $x(t) = te^{-t}U(t)$. Solve the differential equation.
Problem 2.8 Calculate the Laplace transforms of the following functions.

(i) $y(t) = \int_0^t [\delta(t-\tau) - 2e^{-(t-\tau)}U(t-\tau)]\tau^2 e^{-\tau} d\tau,\ t \geq 0,$

(ii) $y(t) = \int_0^t \sin 2(t-\tau)\tau \cos\tau d\tau,\ t \geq 0,$

(iii) $y(t) = \int_{\infty}^{\infty} (t-\tau)^2 \cos(t-\tau)U(t-\tau)e^{-2\tau}U(\tau)d\tau.$