Problems on Matrices

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1 Matrices

Problem 1.1 Given n-th order matrix $A = \begin{bmatrix} a & a & \cdots & a \\ a & a & \cdots & a \\ \cdots & \cdots & \cdots & \cdots \\ a & a & \cdots & a \end{bmatrix}$, calculate A^k.

Problem 1.2 Calculate the n-th powers of the following matrices.

$$\text{(i)} \begin{bmatrix} 0 & 0 & \cdots & 0 & a \\ 0 & 0 & \cdots & a & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 0 & 1 & & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Problem 1.3 Verify the following equations. Note that $a \neq 1$.

$$\text{(i)} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} a^n & \dfrac{(a^n-1)b}{a-1} \\ 0 & 1 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix},$$

$$\text{(iii)} \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & n(n-1)\lambda^{n-2}/2 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix}.$$

Problem 1.4 Given $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ and $f(x) = x^3 - 2x^2 - 9x$, find $f(A)$ and $f'(A)$.

Problem 1.5 A complex matrix A is called a normal matrix if $A^*A = AA^*$. It is called a unitary matrix if $A^*A = AA^* = I$. Verify $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$ $(\omega = (-1+\sqrt{3}i)/2)$ is a normal matrix. Find c such that cA is a unitary matrix.

Problem 1.6 Verify that $\begin{bmatrix} 1/2 & -i/2 & (1-i)/2 \\ i/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/2 & -i/2 & (-1+i)/2 \end{bmatrix}$ is a unitary matrix.

Problem 1.7 Prove that there exists a k such that the following matrices satisfy $A^k = I$.

$$\text{(i)} \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}.$$

Problem 1.8 Find the determinants of the following matrices.

$$\text{(i)} \begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -1 & 2 & 1 \end{vmatrix}, \quad \text{(ii)} \begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & 1 \\ 4 & -2 & 6 \end{vmatrix}, \quad \text{(iii)} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix}, \quad \text{(iv)} \begin{vmatrix} 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & 6 \\ 4 & 1 & 2 & 9 \\ 2 & 4 & 2 & 7 \end{vmatrix}.$$

Problem 1.9 Calculate the determinants of the following matrices.

$$\text{(i)} \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 0 \end{vmatrix}, \quad \text{(ii)} \begin{vmatrix} x & a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_1 & x & a_2 & \cdots & a_{n-1} & 1 \\ a_1 & a_2 & x & \cdots & a_{n-1} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & a_3 & \cdots & x & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n & 1 \end{vmatrix}, \quad \text{(iii)} \begin{vmatrix} x & a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_1 & x & a_2 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & x & \cdots & a_{n-1} & a_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & a_3 & \cdots & x & a_n \\ a_1 & a_2 & a_3 & \cdots & a_n & x \end{vmatrix},$$

Problem 1.10 Show $\begin{vmatrix} 1 & a & b & c+d \\ 1 & b & c & d+a \\ 1 & c & d & a+b \\ 1 & d & a & b+c \end{vmatrix} = 0.$

Problem 1.11 Verify $\begin{vmatrix} a+b & a & \cdots & a \\ a & a+b & \cdots & a \\ \cdots & \cdots & \cdots & \cdots \\ a & a & \cdots & a+b \end{vmatrix} = b^{n-1}(na+b)$ where n is the dimension of the matrix.

Problem 1.12 Suppose that A and B are n-th order square matrices. Verify that the following equations hold.

$$\text{(i)} \begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A+B||A-B|, \quad \text{(ii)} \begin{vmatrix} A & -A \\ B & B \end{vmatrix} = 2^n|A||B|,$$

$$\text{(iii)} \begin{vmatrix} A & -B \\ B & A \end{vmatrix} = |A+iB||A-iB| = |\det(A+iB)|^2.$$

Problem 1.13 Calculate the ranks of the following matrices.

$$\text{(i)} \begin{vmatrix} 2 & -4 & 2 & -3 & 6 \\ -1 & 2 & -5 & -1 & 0 \\ 2 & -4 & -14 & -13 & 18 \\ -5 & 10 & -17 & 0 & -6 \end{vmatrix}, \quad \text{(ii)} \begin{vmatrix} 1 & 2 & -3 & 2 & 1 \\ 1 & 3 & -3 & 2 & 0 \\ 2 & 4 & -6 & 3 & 4 \\ 1 & 1 & -4 & 1 & 6 \end{vmatrix}, \quad \text{(iii)} \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{vmatrix},$$

$$\text{(iv)} \begin{vmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ a_3b_1 & a_3b_2 & \cdots & a_3b_m \\ \cdots & \cdots & \cdots & \cdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{vmatrix} (a_1b_1 \neq 0), \quad \text{(v)} \begin{vmatrix} 1 & a & a & \cdots & a \\ a & 1 & a & \cdots & a \\ a & a & 1 & \cdots & a \\ \cdots & \cdots & \cdots & \cdots & \\ a & a & a & \cdots & 1 \end{vmatrix} (n \text{ 次正方行列}).$$

Problem 1.14 Verify whether the ranks of the following matrices are the same as those of A.

$$\text{(i)} \begin{bmatrix} A & A \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} A & -A \\ -A & A \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \text{(iv)} \begin{bmatrix} A & A^T \end{bmatrix}.$$

Problem 1.15 Obtain the inverse matrices of the following matrices.

$$\text{(i)} \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 2 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 2 & 1 & -1 & 2 \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\text{(iv)} \begin{bmatrix} 0 & 2 & -1 & 4 \\ 1 & 3 & 1 & 0 \\ -2 & 1 & 0 & 2 \\ 2 & 4 & -1 & 2 \end{bmatrix}, \quad \text{(v)} \begin{bmatrix} 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 3 & 0 \\ -2 & -1 & 1 & 2 & 3 \\ 1 & 0 & 1 & -1 & -1 \\ -1 & -2 & -3 & 1 & 0 \end{bmatrix}.$$